

# Triangularization properties of power linear maps and the Structural Conjecture

Michiel de Bondt\*

Department of Mathematics, Radboud University  
Nijmegen, The Netherlands  
*E-mail:* M.deBondt@math.ru.nl

Dan Yan

School of Mathematical Sciences, Graduate University of  
Chinese Academy of Sciences, Beijing 100049, China  
*E-mail:* yan-dan-hi@163.com

February 28, 2013

## Abstract

In this paper, we discuss several additional properties a power linear Keller map may have. The Structural Conjecture by Drużkowski in [Dru] asserts that two such properties are equivalent, but we show that one of this properties is stronger than the other. We even show that the property of linear triangularizability is strictly in between. Furthermore, we give some positive results for small dimensions and small Jacobian ranks.

**Keywords.** Jacobian Conjecture, Drużkowski map, Structural Conjecture, Linearly triangularizable.

**MSC 2010.** 14R10; 14R15.

## 1 Introduction

Throughout this paper, we will write  $K$  for any field of characteristic zero,  $\bar{K}$  for its algebraic closure, and  $K[x] = K[x_1, x_2, \dots, x_n]$  for the polynomial algebra over  $K$  with  $n$  indeterminates  $x = x_1, x_2, \dots, x_n$ . Let  $F = (F_1, F_2, \dots, F_n) : K^n \rightarrow K^n$  be a polynomial map, that is,  $F_i \in K[x]$  for all  $1 \leq i \leq n$ , or briefly

---

\*The first author was supported by the Netherlands Organisation for Scientific Research (NWO).

$F \in K^n$ . We view  $F$  and  $x$  as column matrices, as well as  $\partial = \partial_1, \partial_2, \dots, \partial_n$ , where  $\partial_i = \frac{\partial}{\partial x_i}$ . Let  $M^t$  be the transpose of a matrix  $M$  and write

$$\mathcal{J}F = (\partial(F^t))^t = \begin{pmatrix} \partial_1 F_1 & \partial_2 F_1 & \cdots & \partial_n F_1 \\ \partial_1 F_2 & \partial_2 F_2 & \cdots & \partial_n F_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_1 F_n & \partial_2 F_n & \cdots & \partial_n F_n \end{pmatrix}$$

We say that a polynomial map  $F$  is a Keller map if  $\det \mathcal{J}F \in K^*$ . The well-known Jacobian Conjecture, raised by O.H. Keller in 1939 in [Kel], states that a polynomial map  $F : K^n \rightarrow K^n$  is invertible if it is a Keller map. This conjecture is still open for all  $n \geq 2$ . In [Dru, Th. 3], Ludwik Drużkowski showed that it suffices to consider polynomial maps  $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$  of the form  $F = x + (Ax)^{*3}$ , where  $A \in \text{Mat}_n(\mathbb{C})$  and  $M^{*d}$  is the  $d$ -th Hadamard power (repeated Hadamard product with itself) of a matrix  $M$ .

In the same paper, Drużkowski also formulated the Structural Conjecture, which asserts the following. Write  $M|_{x=G}$  for the substitution of  $x$  by  $G$  in a matrix  $M$ .

**Structural Conjecture.** *If  $F = x + (Ax)^{*3}$  and  $\det \mathcal{J}F = 1$ , then the following conditions are equivalent.*

(JC)  $\det((\mathcal{J}F)|_{x=v_1} + (\mathcal{J}F)|_{x=v_2}) \neq 0$  for all  $v_1, v_2 \in \mathbb{C}^n$ .

(\*\*) *There exists  $b_i \in \mathbb{C}^{n-1}$  and  $c_j \in \mathbb{C}^{n-1}$  such that  $c_j^t b_i = 0$  for every  $i \geq j \geq 1$ , and  $F$  has the form  $x + \sum_{i=1}^{n-1} (c_i^t x)^d b_i$ .*

Actually, Drużkowski writes  $F = x + \sum_{j=1}^n (a_j^t x)^3 e_j$  instead of  $F = x + (Ax)^{*3}$ , where  $e_j$  is the  $j$ -th standard basis unit vector. Hence  $a_j^t$  corresponds to the  $j$ -th row  $A_j$  of  $A$ . Since the vectors  $c_j$  and  $b_i$  are viewed as column matrices, the matrix product  $c_j^t b_i$  has only one entry, which we see as an element of  $\mathbb{C}$ .

We call a polynomial map  $F$  over  $K$  *linearly triangularizable* if there exists a  $T \in \text{GL}_n(K)$  such that the Jacobian of  $T^{-1}F(Tx)$  is a triangular matrix. For Keller maps of the form  $F = x + H$  with  $H$  homogeneous of degree  $d \geq 2$ , the existence of such a  $T$  automatically means that the diagonal of

$$\mathcal{J}(T^{-1}F(Tx)) = T^{-1}(\mathcal{J}H)|_{x=Tx}T$$

is zero, because  $\mathcal{J}H$  has to be nilpotent due to the Keller condition.

We embed the Structural Conjecture in a more general scope, where  $F$  has the form  $x + H$  such that  $\mathcal{J}H$  is nilpotent, and compare its conditions with linear triangularizability and other properties. We give positive results in special cases and counterexamples in general. When we give counterexamples, we will give one of the form  $F = x + H$  with  $H$  homogeneous of degree  $d$  and one of the form  $F = x + (Ax)^{*d}$ , for every  $d \geq 3$ .

## 2 Triangularization

In the following proposition, the conditions (JC) and (\*\*) of the Structural Conjecture are included in a chain of six properties. Furthermore, we generalize to maps  $x \mapsto H$  such that  $H$  has no constant terms instead of being homogeneous.

**Proposition 2.1.** *Let  $F = x + H$  be any polynomial map of degree  $d$  over  $K$ . Then for*

$$F \text{ is invertible} \quad (\text{JC}^-)$$

$$\det \left( \sum_{i=1}^{d-1} \mathcal{J}F|_{x=v_i} \right) \in \bar{K}^* \text{ for all } v_i \in \bar{K}^n \quad (\text{JC})$$

$$\det \left( \sum_{i=1}^n \mathcal{J}F|_{x=v_i} \right) \in \bar{K}^* \text{ for all } v_i \in \bar{K}^n \quad (\text{JC}^+)$$

and for the existence of  $b_i \in K^n$ ,  $c_j \in K^n$  and  $d_i \in \{1, 2, \dots, d\}$  such that  $c_j^\dagger b_i = 0$  for every  $i \geq j \geq 1$ , and

$$H = \sum_{i=1}^N (c_i^\dagger x)^{d_i} b_i \text{ for some } N \in \mathbb{N} \quad (*)$$

$$H = \sum_{i=1}^{n-1} (c_i^\dagger x)^{d_i} b_i \quad (**)$$

$$H = \sum_{i=1}^{n-1} (c_i^\dagger x)^{d_i} b_i \text{ and } b_1, b_2, \dots, b_{n-1} \text{ are independent} \quad (***)$$

we have  $(\text{JC}^-) \Leftarrow (\text{JC}) \Leftarrow (\text{JC}^+) \Leftarrow (*) \Leftarrow (**) \Leftarrow (***)$ .

Furthermore,  $(\text{JC}^-)$ ,  $(\text{JC})$  and  $(\text{JC}^+)$  are satisfied when  $d \leq 2$ .

*Proof.* Notice that the last two implications are trivial. The first two implications follow from [GdBDS, Cor. 2.3] and [GdBDS, Th. 3.5] respectively. The last claim follows from the first two implications and [GdBDS, Prop. 3.1].

To show the third implication, assume that  $(*)$  holds. Then

$$S := \sum_{k=1}^n (\mathcal{J}H)|_{x=v_k} = \sum_{k=1}^n \sum_{i=1}^N b_i \cdot d_i (c_i^\dagger v_k)^{d_i-1} \cdot c_i^\dagger = \sum_{i=1}^N b_i \left( d_i \sum_{k=1}^n (c_i^\dagger v_k)^{d_i-1} \right) c_i^\dagger$$

It follows that in the expansion of  $S^{N+1}$ , each term will have a factor  $c_j^\dagger \cdot b_i$  such that  $i \geq j \geq 1$ , which is zero by assumption. Hence  $S^{N+1} = 0$ . Thus  $S$  is nilpotent and  $\det(\sum_{i=1}^n \mathcal{J}F|_{x=v_i}) = \det(nI_n + S) = n^n \neq 0$ .  $\square$

In the last section, we will show that  $(\text{JC}^-) \not\Rightarrow (\text{JC})$  and  $(\text{JC}^+) \not\Rightarrow (*) \not\Rightarrow (**) \not\Rightarrow (***)$ , even when  $H$  is *homogeneous power linear*, i.e.  $H = \sum_{i=1}^n (c_i^\dagger x)^{d_i} e_i$  for some  $c_i \in K^n$  and a  $d \geq 1$ . But first, we formulate a lemma and a theorem about the starred equations. We call  $H$  *non-homogeneous power linear* if  $H = \sum_{i=1}^n (c_i^\dagger x)^{d_i} e_i$  for some  $c_i \in K^n$  and some  $d_i \geq 1$ .

**Lemma 2.2.** Let  $N \in \mathbb{N}$  and suppose that there exist  $b_i \in K^n$ ,  $c_j \in K^n$  and  $d_i \in \mathbb{N}$  such that

$$H = \sum_{i=1}^N (c_i^t x)^{d_i} b_i$$

Then the following statements are equivalent.

- i) There exists a  $\sigma \in S_N$  such that  $c_{\sigma(j)}^t b_{\sigma(i)} = 0$  for every  $i \geq j \geq 1$ .
- ii) There exists a  $T \in \text{GL}_n(K)$  such that the Jacobian of  $T^{-1}(c_i^t T x)^{d_i} b_i$  is lower triangular with zeroes on the diagonal for all  $i \leq N$ .

Furthermore, if  $b_1, b_2, \dots, b_N$  are independent and i) holds, then we can choose  $\sigma$  such that

$$b_{\sigma(i)} = T e_{n-N+i} \quad (1)$$

for each  $i$ .

*Proof.* We prove that i) and ii) are equivalent, showing the last claim along the road.

- ii)  $\Rightarrow$  i) Suppose that ii) holds. Let  $m_j$  be the number of trailing coordinates zero of  $T^t c_j$  for each  $j$ . By reordering terms of  $H$ , we can obtain that  $m_j \geq m_i$  for each  $i \geq j \geq 1$ . By ii),

$$\mathcal{J}(T^{-1}(c_i^t T x)^{d_i} b_i) = T^{-1} b_i \cdot d_i (c_i^t T x)^{d_i-1} \cdot c_i^t T \quad (2)$$

is lower triangular with zeroes on the diagonal. Hence the number of leading coordinates zero of  $T^{-1} b_i$  is at least  $n - m_i \geq n - m_j$  for each  $i \geq j \geq 1$ . Comparing the numbers of leading and trailing coordinates zero, we get  $c_j^t b_i = c_j^t T \cdot T^{-1} b_i = 0$  for all  $i \geq j \geq 1$ , which is i) with  $\sigma = 1$ . So we can take  $\sigma = 1$  when  $m_j \geq m_{j+1}$  for each  $j$  a priori.

- i)  $\Rightarrow$  ii) Suppose that i) holds. Again by reordering terms of  $H$ , we can obtain that  $\sigma = 1$ . Suppose that the matrix with columns  $b_1, b_2, \dots, b_N$  has rank  $r$ . Then there are  $\tau(1) < \tau(2) < \dots < \tau(r)$  such that  $b_{\tau(r)} \neq 0$  and  $b_{\tau(k)}$  is independent of  $b_{\tau(k+1)}, b_{\tau(k+2)}, \dots, b_{\tau(r)}$  for each  $k < r$ . By definition of  $r$ ,  $b_i$  is dependent of  $b_{\tau(k)}, b_{\tau(k+1)}, \dots, b_{\tau(r)}$  for all  $k$  and all  $i > \tau(k-1)$ , where  $\tau(0) = 0$ . Furthermore,  $c_i b_{\tau(k)} = c_i b_{\tau(k+1)} = \dots = c_i b_{\tau(r)} = 0$  for all  $k$  and all  $i \leq \tau(k)$  on account of i) with  $\sigma = 1$ .

Take  $T \in \text{GL}_n(K)$  such that the last  $r$  columns of  $T$  are  $b_{\tau(1)}, b_{\tau(2)}, \dots, b_{\tau(r)}$ , in that order. Then we have (1) with  $\sigma = 1$  in case  $b_1, b_2, \dots, b_N$  are independent. Take  $i \leq N$  arbitrary. It suffices to show that (2) is lower triangular with zeroes on the diagonal. This is trivial when  $b_i = 0$ , so assume that  $b_i \neq 0$ . Then by definition of  $r$  and  $\tau$ , there exists a  $k \geq 1$  such that  $\tau(k) \geq i > \tau(k-1)$ . As we have seen above,  $b_i$  is dependent of  $b_{\tau(k+1)}, b_{\tau(k+2)}, \dots, b_{\tau(r)}$  and  $c_i b_{\tau(k)} = c_i b_{\tau(k+1)} = \dots = c_i b_{\tau(r)} = 0$ .

Hence  $T^{-1} b_i$  is dependent of  $e_{n-r+k}, e_{n-r+k+1}, \dots, e_n$  and  $c_i^t T e_{n-r+k} = c_i^t T e_{n-r+k+1} = c_i^t T e_n = 0$  by definition of  $T$ . Consequently, all nonzero

entries of (2) are within the submatrix consisting of rows  $n-r+k, n-r+k+1, \dots, n$  and columns  $1, 2, \dots, n-r+k-1$  of it. Since  $i$  was arbitrary, we obtain ii).  $\square$

**Theorem 2.3.** *Let  $x+H$  be any map of degree  $d \geq 1$  over  $K$  such that  $H(0) = 0$ . Then we have the following.*

- i)  *$H$  is of the form (\*), if and only if there exists a  $T \in \text{GL}_n(K)$  such that the Jacobian of  $T^{-1}H(Tx)$  is lower triangular with zeroes on the diagonal, i.e.  $H$  is linearly triangularizable and  $\mathcal{J}H$  is nilpotent.*
- ii)  *$H$  is of the form (\*\*), if and only if there exists  $b_i, c_j \in K^n$  and there exists a  $T \in \text{GL}_n(K)$  such that  $H = \sum_{i=1}^{n-1} (c_i^t x)^{d_i} b_i$  and the Jacobian of  $T^{-1}(c_i^t Tx)^{d_i} b_i$  is lower triangular with zeroes on the diagonal for all  $i \leq n-1$ .*
- iii)  *$H$  is of the form (\*\*\*), if and only if there exists a  $T \in \text{GL}_n(K)$  such that each component of  $T^{-1}H(Tx)$  is a power of a linear form and the Jacobian of  $T^{-1}H(Tx)$  is lower triangular with zeroes on the diagonal.*

*Proof.* Since the three results have similarities, we structure the proof as follows.

**Only-if-parts.** All only-if-parts follow immediately from i)  $\Rightarrow$  ii) in lemma 2.2, except the claim that each component of  $T^{-1}H(Tx)$  is a power of a linear form in iii). So assume that  $H$  is of the form (\*\*\*). By (\*\*) and (1), we have that

$$T^{-1}H(Tx) = \sum_{i=1}^{n-1} T^{-1}(c_{\sigma(i)}^t Tx)^{d_i} b_{\sigma(i)} = \sum_{i=1}^{n-1} (c_{\sigma(i)}^t Tx)^{d_i} e_{i+1}$$

for some  $\sigma \in S_{n-1}$ . So  $T^{-1}H(Tx)$  is of the desired form.

**If-parts.** The if-part of ii) follows immediately from ii)  $\Rightarrow$  i) of lemma 2.2. To prove the if-part of i), suppose that  $T^{-1}H(Tx)$  has a lower triangular Jacobian with zeroes on the diagonal. Then there exists an  $r \in \mathbb{N}$  such that we can write the  $(i+1)$ -th component of  $T^{-1}H(Tx)$  as a linear combination of  $r$  powers of linear forms  $c_j^t Tx$  in  $K[x_1, x_2, \dots, x_i]$  for each  $i \geq 1$ . Furthermore, the first component of  $T^{-1}H(Tx)$  is zero on account of  $H(0) = 0$ . Hence we have

$$\begin{aligned} T^{-1}H(Tx) &= \sum_{i=1}^{n-1} \sum_{j=1}^r (c_{r(i-1)+j}^t Tx)^{d_{r(i-1)+j}} e_{i+1} \\ &= T^{-1} \sum_{i=1}^{n-1} \sum_{j=1}^r (c_{r(i-1)+j}^t Tx)^{d_{r(i-1)+j}} T e_{i+1} \end{aligned}$$

Taking  $b_{r(i-1)+j} = T e_{i+1}$  for all  $i$  and for all  $j$  with  $1 \leq j \leq r$ , we have

$$H = \sum_{i=1}^{n-1} \sum_{j=1}^r (c_{r(i-1)+j}^t x)^{d_{r(i-1)+j}} b_{r(i-1)+j} = \sum_{i=1}^{r(n-1)} (c_i^t x)^{d_i} b_i$$

Furthermore, for each  $j$ , the Jacobian of  $(c_j^t T x)^{d_j} T^{-1} b_j$  only has nonzero entries within the submatrix consisting of row  $i+1$  and columns  $1, 2, \dots, i$  of it by definition of  $c_j$  and  $b_j$ , where  $i = \lceil j/r \rceil$ . Hence the Jacobian of  $(c_j^t T x)^{d_j} T^{-1} b_j$  is lower triangular with zeroes on the diagonal for all  $j$ . Now the if-part of i) follows from ii)  $\Rightarrow$  i) of lemma 2.2. The if-part of iii) follows as well, because we have  $r = 1$  and additionally that the  $b_i$ 's are independent in that case.  $\square$

### 3 Positive results

First, we formulate a theorem about maps  $x + H$  such that  $H$  is homogeneous and  $\mathcal{J}H$  is nilpotent.

**Theorem 3.1.** *Assume that  $H \in K[x]^n$  is homogeneous of degree  $d \geq 1$ , such that  $\mathcal{J}H$  is nilpotent. Then we have (\*\*\*) (and hence five  $\Rightarrow$ 's) in case  $n \leq 2$ , and (\*) (and hence three  $\Rightarrow$ 's) in case  $n = 3$  or  $n = 4 = d + 2$ . Furthermore, the implication chain  $(\text{JC}) \Rightarrow (\text{JC}^+) \Rightarrow (*)$  holds when  $n = 4 = d + 1$ .*

*If  $H$  is power linear in addition, then the above claims even hold when we replace the estimates on  $n$  by estimates on  $\text{rk } \mathcal{J}H$ .*

*Proof.* We show the equivalent properties in i) and iii) of theorem 2.3 respectively instead of (\*) and (\*\*\*). We start with the cases where  $H$  is only homogeneous.

The case  $n \leq 2$  follows from [Che, Lem. 3] and the case  $n = 3$  follows from [dBvdE, Th. 1.1]. The case  $n = 4 = d + 2$  follows from a corresponding strong nilpotence result in [MO], and the equivalence of strong nilpotence and the property in i) of theorem 2.3, which is proved in [vdEH]. The case  $n = 4 = d + 1$  follows from [dB1, Th. 4.6.5] and the fact that  $F = x + H$ , with  $H$  as in [dB1, Th. 4.6.5], does not satisfy (JC), because the rightmost two columns of  $(\mathcal{J}F)|_{x=(1,i,0,0)} + (\mathcal{J}F)|_{x=(1,-i,0,0)}$  are equal, where  $H = (0, \lambda x_1^3, x_2(x_1 x_3 - x_2 x_4) + p(x_1, x_2), x_1(x_1 x_3 - x_2 x_4) + q(x_1, x_2))$ .

Assume from now on that  $H$  is power linear in addition. The case  $\text{rk } \mathcal{J}H \leq 2$  follows from [TdB, Th. 4.7], because  $K = \mathbb{C}$  is not used in its proof, or theorem 3.2 below. The cases  $\text{rk } \mathcal{J}H = 3$  and  $\text{rk } \mathcal{J}H = 4 = d + 2$  follow by way of [Che, Th. 2] from the cases  $n = 3$  and  $n = 4 = d + 2$  respectively. The case  $\text{rk } \mathcal{J}H = 4 = d + 1$  follows from the case  $n = 4 = d + 1$  by way of a variant of [Che, Th. 2], namely with (\*) replaced by  $(\text{JC}) \Rightarrow (*)$ . To prove this variant, one can follow the proof of [Che, Th. 2], to see that it suffices to show that  $F_1 = T^{-1} \circ F \circ T$  satisfies (JC) in case  $F$  does, in that proof.  $\square$

In theorem 3.2 below, which is the non-homogeneous variant of theorem 3.1, we must replace the estimates on  $n$  and  $\text{rk } \mathcal{J}H$  of theorem 3.1 by estimates on  $n + 1$  and  $\text{rk } \mathcal{J}H + 1$  respectively, except the estimate  $n \leq 2$  for  $(**) \Rightarrow (***)$ , and the estimate  $\text{rk } \mathcal{J}H \leq 2$  for  $(***)$ , which can be maintained.

**Theorem 3.2.** *Assume that  $H \in K[x]^n$  has degree  $d$ , such that  $H(0) = 0$  and  $\mathcal{J}H$  is nilpotent. Then we have (\*\*\*) (and hence five  $\Rightarrow$ 's) in case  $n \leq 1$ ,*

both  $(*)$  and  $(**) \Rightarrow (***)$  (and hence four  $\Rightarrow$ 's) in case  $n = 2$ , and  $(*)$  (and hence three  $\Rightarrow$ 's) in case  $n = 3 = d + 1$ . Furthermore, the implication chain  $(JC) \Rightarrow (JC^+) \Rightarrow (*)$  holds when  $n = 3 = d$ .

If  $H$  is power linear in addition, then the above claims even hold when we replace the estimates on  $n$  by estimates on  $\text{rk } \mathcal{J}H$ , and additionally  $(***)$  (and hence five  $\Rightarrow$ 's) holds when  $\text{rk } \mathcal{J}H = 2$ .

Furthermore, if we replace  $(*)$  and  $(***)$  by their equivalences in i) and iii) of theorem 2.3, then the condition  $H(0) = 0$  is no longer necessary.

*Proof.* We show the equivalent properties in i) and iii) of theorem 2.3 respectively instead of  $(*)$  and  $(***)$ . We start with the cases where  $H$  only has a nilpotent Jacobian.

The case  $n = 1$  is trivial, because  $H = 0$  in that case. Notice that in the cases  $n = 2$  and  $n = 3 = d + 1$ , the homogeneization  $x_{n+1}^d H(x_{n+1}^{-1}x, 0)$  of  $H$  has a strongly nilpotent Jacobian on account of theorem 3.1. By substituting  $x_{n+1} = 1$ , we see that the Jacobian of  $H$  itself is strongly nilpotent as well. By the equivalence of strong nilpotence and the property in i) of theorem 2.3, which is proved in [vdEH], we have the property in i) of theorem 2.3, and hence also  $(*)$ , when  $n = 2$  or  $n = 3 = d + 1$ . This gives the case  $n = 3 = d + 1$ , and also the case  $n = 2$ , because  $(**)$  and  $(***)$  are trivially equivalent when  $n = 2$ .

In order to prove the case  $n = 3 = d$ , assume that  $H$  does not have the property in i) of theorem 2.3. By [dB1, Cor. 4.6.6], we may assume that the first components of  $T^{-1}H(Tx)$  equals  $\lambda \in K$  for some  $T \in \text{GL}_3(K)$ . Following the proof of [dB1, Th. 4.6.5], we see that  $T^{-1}H(Tx) = (\lambda, x_1(x_2 - x_1x_3) + p(x_1), (x_2 - x_1x_3) + q(x_1))$  for some  $T \in \text{GL}_3(K)$ . Since the rightmost two columns of  $(\mathcal{J}F)|_{x=(i,0,0)} + (\mathcal{J}F)|_{x=(-i,0,0)}$  are equal, we see that  $(JC)$  does not hold, as desired.

Assume from now on that  $H$  is (non-homogeneous) power linear in addition. The cases  $\text{rk } \mathcal{J}H = 3 = d$  and  $\text{rk } \mathcal{J}H = 3 = d + 1$  follow in a similar manner as the cases  $\text{rk } \mathcal{J}H = 4 = d + 1$  and  $\text{rk } \mathcal{J}H = 4 = d + 2$  respectively in theorem 3.1. So assume that  $\text{rk } \mathcal{J}H \leq 2$ . Take  $\lambda$  and  $\mu$  as in lemma 3.3 below. If  $\mu^t H$  is a power of a linear form, then we take  $T \in \text{GL}_n(K)$  such that  $\lambda^t$  and  $\mu^t$  are the first two rows of  $T^{-1}$ , in that order, and for the remaining rows of  $T^{-1}$  we transpose standard basis unit vectors. Since  $\lambda^t$  and  $\mu^t$  generate the row space of  $\mathcal{J}H$ , we see that  $\lambda^t T = e_1$  and  $\mu^t T = e_2$  generate the row space of  $\mathcal{J}(T^{-1}H(Tx))$ . Using additionally that  $\mu^t H \in K[\lambda^t x]$ , we obtain that  $T^{-1}H(Tx) \in K \times K[x_1] \times K[x_1, x_2]^{n-2}$  has a lower triangular Jacobian with zeroes on the diagonal. So we have i) of theorem 2.3 and hence also  $(*)$ .

So assume that  $\mu^t H$  is not a power of a linear form. By lemma 3.3 below, we have

$$\mu^t H = \nu_1(\lambda^t x)^{d_1} + \nu_2(\lambda^t x)^{d_2} + \cdots + \nu_r(\lambda^t x)^{d_r} \quad (3)$$

where  $r \geq 2$ ,  $\nu \in (K \setminus \{0\})^r$  and  $\{0, 1\} \ni \lambda^t H \leq d_1 < d_2 < \cdots < d_r$ . Take  $T \in \text{GL}_n(K)$  such that  $\lambda^t$  is the first row of  $T^{-1}$ . Let  $V \in \text{Mat}_{r,n}(\{0, 1\})$  such that  $V_{ij} = 1$ , if and only if  $\deg H_i = d_j$ . Without worrying about independence of rows at this stage, take for each  $i$  with  $2 \leq i \leq r$ , the  $(i+1)$ -th row of  $T^{-1}$  equal to  $T_{i+1}^{-1} = \nu_i^{-1} \mu^t * V_i$ , where  $*$  is the Hadamard product and  $V_i$  is the  $i$ -th row

of  $V$ . Then by definition of  $V_i$ , the  $(i+1)$ -th component  $T_{i+1}^{-1}H$  of  $T^{-1}H$  is  $\nu_i^{-1}$  times the homogeneous part of degree  $d_i$  of (3), which is  $\nu_i^{-1}\nu_i(\lambda^t x)^{d_i} = (\lambda^t x)^{d_i}$ , for each  $i$  with  $2 \leq i \leq r$ .

Still without worrying about independence of rows, take the second row of  $T^{-1}$  equal to

$$T_2^{-1} = \nu_1^{-1}(\mu^t - (\mu^t * (V_2 + \cdots + V_r)))$$

Since  $\mu^t * V_i = \nu_i T_{i+1}^{-1}$  for each  $i$  with  $2 \leq i \leq r$  by definition of  $T^{-1}$ , we have

$$T_2^{-1} = \nu_1^{-1}(\mu^t - (\nu_2 T_3^{-1}H + \cdots + \nu_r T_{r+1}^{-1}H)) \quad (4)$$

and the second component of  $T^{-1}H$  equals

$$\begin{aligned} T_2^{-1}H &= \nu_1^{-1}(\mu^t H - (\nu_2 T_3^{-1}H + \cdots + \nu_r T_{r+1}^{-1}H)) \\ &= \nu_1^{-1}(\mu^t H - (\nu_2 (\lambda^t x)^{d_2} + \cdots + \nu_r (\lambda^t x)^{d_r})) \end{aligned}$$

which by (3) is equal to  $\nu_1^{-1}\nu_1(\lambda^t x)^{d_1} = (\lambda^t x)^{d_1}$ . Thus for each  $i \in \{1, 2, \dots, r\}$ , the  $(i+1)$ -th component of  $T^{-1}H$  is equal to  $(\lambda^t x)^{d_i}$ .

Since  $\{0, 1\} \ni \lambda^t H \leq d_1 < d_2 < \cdots < d_r$ , we have  $\deg T_1^{-1}H = \deg \lambda^t H < \deg \mu^t H$  and the degrees of the first  $r+1$  components of  $T^{-1}H$  are strictly increasing. Hence by  $T_1^{-1} = \lambda^t \neq 0$ , the first  $r+1$  rows of  $T^{-1}$  are indeed independent. Take transposed standard basis unit vectors for the remaining rows of  $T^{-1}$ . By  $\lambda^t T = e_1^t$ , we get that the first  $r+1$  components of  $T^{-1}H(Tx)$  are  $\lambda^t H, x_1^{d_1}, x_1^{d_2}, \dots, x_1^{d_r}$ , so  $T^{-1}H(Tx) \in K \times K[x_1]^r \times K[x]^{n-r-1}$ . Furthermore, we see that  $T^{-1}H(Tx)$  is power linear.

By (4), we have  $\sum_{i=1}^r \nu_i T_{i+1}^{-1} = \mu^t$ , so  $\mu^t$  is a linear combination of the first  $r+1$  rows of  $T^{-1}$ . Since  $\lambda^t$  and  $\mu^t$  generate the row space of  $\mathcal{J}H$  and are linear combinations of the first  $r+1$  rows of  $T^{-1}$ , we see that  $\lambda^t T$  and  $\mu^t T$  generate the row space of  $\mathcal{J}T^{-1}H(Tx)$  and are linear combinations of  $e_1^t, e_2^t, \dots, e_{r+1}^t$ . Hence  $T^{-1}H(Tx) \in K[x_1, x_2, \dots, x_{r+1}]^n$ . By taking intersection,  $T^{-1}H(Tx) \in K \times K[x_1]^r \times K[x_1, x_2, \dots, x_{r+1}]^{n-r-1}$  has a lower triangular Jacobian with zeroes on the diagonal. So we have i) of theorem 2.3 and hence also (\*).  $\square$

**Lemma 3.3.** *Assume that  $H \in K[A_1x, A_2x, \dots, A_nx]^n$ , where  $A_j$  is the  $j$ -th row of a matrix  $A \in \text{Mat}_n(K)$  such that  $\text{rk } A \leq 2$  and  $\mathcal{J}H$  is nilpotent. Then there exists independent  $\lambda, \mu \in K^n$ , such that  $\mu^t H \in K[\lambda^t x]$  has no terms of degree less than  $\lambda^t H \in \{0, 1\}$ , and  $\lambda^t$  and  $\mu^t$  generate the row space of  $A$ .*

*Proof.* Using the case  $n = 2$  of theorem 3.2 (instead of the case  $n = 3$  of theorem 3.1), we obtain by similar techniques as in the proof of the case  $\text{rk } \mathcal{J}H = 3$  of theorem 3.1 that there exists a  $T \in \text{GL}_n(K)$  such that  $ATx \in K[x_1, x_2]^n$  and the Jacobian of  $T^{-1}H(Tx) \in K[x_1, x_2]^n$  is lower triangular with zeroes on the diagonal. By a subsequent linear conjugation on the first two coordinates, we can even obtain that additionally, the first component of  $T^{-1}H(Tx)$  is contained in  $\{0, 1\}$  and that the second component of  $T^{-1}H(Tx)$  has no constant term in case the first component already has.

Now take for  $\lambda^t$  the first row of  $T^{-1}$  and for  $\mu^t$  the second row of  $T^{-1}$ . Then  $\lambda^t H(Tx) \in \{0, 1\}$  and  $ATx \in K[x_1, x_2]^n$ . Furthermore,  $\mu^t H(Tx) \in K[x_1]$  only



has terms of degree greater than  $\deg \lambda^t H(Tx)$ , and hence no terms of degree less than  $\lambda^t H(Tx)$  itself. Thus substituting  $x = T^{-1}x$  gives the desired results.  $\square$

Notice that in case  $H$  is power linear and  $\text{rk } \mathcal{J}H = 1$  in theorem 3.2, we can even get  $T^{-1}H(Tx) \in k[x_1]^n$  in iii) of theorem 2.3, namely by taking  $\lambda^t$  in the row space of  $\mathcal{J}H$ . This is similar to the case that  $H$  is power linear and  $\text{rk } \mathcal{J}H = 2$  in theorem 3.1, in the proof of which  $T$  is taken such that  $T^{-1}H(Tx) \in k[x_1, x_2]^n$  in iii) of theorem 2.3. It is however not always possible to take  $T$  such that  $T^{-1}H(Tx) \in k[x_1, x_2]^n$  in iii) of theorem 2.3 when  $H$  is power linear and  $\text{rk } \mathcal{J}H = 2$  in theorem 3.2, which the reader may show by taking e.g.  $H = (0, x_1^d, x_1^{d-1}, (x_2 + x_3)^d)$ .

Theorems 3.1 and 3.2 contain positive results with estimates on  $\text{rk } \mathcal{J}H$ , but for power linear  $H$  only. Theorem 3.4 below however comprises two results with estimates on  $\text{rk } \mathcal{J}H$ , without the requirement that  $H$  is power linear. Furthermore, the homogeneous counterexamples (9) and (10) later in this article show that the estimates in theorem 3.4 cannot be improved, even if we have the extra condition that  $H$  is homogeneous.

**Theorem 3.4.** *Assume that  $H \in K[x]^n$  has degree  $d$ , such that  $H(0) = 0$  and  $\mathcal{J}H$  is nilpotent. If  $\text{rk } \mathcal{J}H = 1$  or  $\text{rk } \mathcal{J}H = 2 = d$ , then  $H$  is of the form (\*).*

*Furthermore, if we replace (\*) by its equivalent in i) of theorem 2.3, then the condition  $H(0) = 0$  is no longer necessary.*

*Proof.* We show the equivalent property in i) of theorem 2.3 instead of (\*). The case  $\text{rk } \mathcal{J}H \leq 1$  follows from a corresponding strong nilpotence result in (2)  $\Rightarrow$  (3) of [dB2, Th. 4.2], and the equivalence of strong nilpotence and the property in i) of theorem 2.3, which is proved in [vdEH].

So assume that  $\text{rk } \mathcal{J}H = 2 = d$  and suppose without loss of generality that  $H(0) = 0$ . The additional claim that the diagonal is zero follows from the nilpotency of  $\mathcal{J}H$ , so we do not need to worry about that any more. By lemma 3.5 below, there exists a  $T \in \text{GL}_n(K)$  such that for  $\tilde{H} := T^{-1}H(Tx)$ , we have one of the following cases, which we treat individually.

- $\tilde{H} \in K[x_1, x_2]^n$ .  
Then by theorem 3.2,  $(\tilde{H}_1, \tilde{H}_2)$  has the property in i) of theorem 2.3. Hence we can choose  $T$  such that  $\mathcal{J}_{x_1, x_2}(\tilde{H}_1, \tilde{H}_2)$  is lower triangular. It follows that  $\mathcal{J}\tilde{H}$  is lower triangular as well, which is the property in i) of theorem 2.3.
- $\tilde{H}_3 = \tilde{H}_4 = \dots = \tilde{H}_n = 0$ .  
Then by [vdE, Th. 7.2.25], we have

$$(\tilde{H}_1, \tilde{H}_2) = (bg(ax_1 - bx_2) + d, ag(ax_1 - bx_2) + c)$$

where  $a, b, c, d \in K[x_3, x_4, \dots, x_n]$  and  $g$  is a univariate polynomial over  $K[x_3, x_4, \dots, x_n]$ . Hence  $a\tilde{H}_1 - b\tilde{H}_2 \in K[x_3, x_4, \dots, x_n]$ . Using that  $\deg(\tilde{H}_1, \tilde{H}_2) = 2$ , we see that either  $g$  or  $ab$  is constant.

In both cases, there exists a nontrivial  $K$ -linear combination of  $\tilde{H}_1$  and  $\tilde{H}_2$  which is contained in  $K[x_3, x_4, \dots, x_n]$ . By choosing  $T$  appropriate, we can get  $\tilde{H}_2 \in K[x_3, x_4, \dots, x_n]$ , in which case  $\mathcal{J}\tilde{H}$  is upper triangular. By a subsequent conjugation of  $\tilde{H}$  with the map  $(x_n, x_{n-1}, \dots, x_2, x_1)$ , we get the desired lower triangular form of the Jacobian, which gives the property in i) of theorem 2.3.

- $\tilde{H}_2 = \tilde{H}_3^2 \neq 0$  and  $\tilde{H}_4 = \tilde{H}_5 = \dots = \tilde{H}_n = 0$ .  
If  $\tilde{H}_3 \in K[x_4, x_5, \dots, x_n]$ , then  $\mathcal{J}\tilde{H}$  is upper triangular, and a subsequent conjugation of  $\tilde{H}$  with the map  $(x_n, x_{n-1}, \dots, x_2, x_1)$  gives the desired result. So assume that  $\tilde{H}_3 \notin K[x_4, x_5, \dots, x_n]$ . Using polynomial extension of scalars on the case  $n = 3$  of theorem 3.2, it follows that there exists a  $\tilde{T} \in \text{GL}_3(K(x_4, x_5, \dots, x_n))$  such that  $\mathcal{J}_{\tilde{x}}(\tilde{T}^{-1}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)|_{\tilde{x}=\tilde{T}\tilde{x}})$  is lower triangular with zeroes on the diagonal, where  $\tilde{x} = x_1, x_2, x_3$ .

By clearing denominators in the first row of  $\tilde{T}^{-1}$ , we see that there exists a  $\lambda \in K[x_4, x_5, \dots, x_n]^3$  such that  $\lambda_1 \tilde{H}_1 + \lambda_2 \tilde{H}_2 + \lambda_3 \tilde{H}_3 \in K[x_4, x_5, \dots, x_n]$ . Since  $\tilde{H}_2$  and  $\tilde{H}_3$  have different positive degrees with respect to  $\tilde{x}$ , it follows that  $\lambda_1 \neq 0$  and that  $\tilde{H}_1 \in K[\tilde{H}_3, x_4, x_5, \dots, x_n]$ .

Now take  $S \in \text{GL}_n(K)$  such that the  $i$ -th row of  $S^{-1}$  equals  $e_i^t$  for all  $i \geq 4$  and the third row of  $S^{-1}$  equals  $\mathcal{J}\tilde{H}_3$ . Then only the first three components of  $S^{-1}\tilde{H}(Sx)$  are nonzero, and we have  $\tilde{H}_3(Sx) = x_3$  and  $(Sx)_i = x_i$  for all  $i \geq 4$ . Consequently, the first three components of  $S^{-1}\tilde{H}(Sx)$  are contained in  $K[x_3, x_4, x_5, \dots, x_n]$ . Hence the Jacobian of  $S^{-1}\tilde{H}(Sx)$  is upper triangular, and a subsequent conjugation of  $\tilde{H}$  with the map  $(x_n, x_{n-1}, \dots, x_2, x_1)$  gives the desired lower triangular form of the Jacobian, and the property in i) of theorem 2.3 follows.  $\square$

**Lemma 3.5.** *Assume that  $H \in K[x]^n$  has degree 2, such that  $H(0) = 0$  and  $\text{rk } \mathcal{J}H \leq 2 = d$ . Then there exists a  $T \in \text{GL}_n(K)$  such that  $\tilde{H} := T^{-1}H(Tx)$  has one of the three forms that are specified in the proof of theorem 3.4.*

*Proof.* We can choose  $T$  such that  $\tilde{H}_1, \tilde{H}_2, \dots, \tilde{H}_r$  have independent quadratic parts,  $\tilde{H}_{r+1}, \tilde{H}_{r+2}, \dots, \tilde{H}_s$  are independent linear forms, and  $\tilde{H}_{s+1} = \tilde{H}_{s+2} = \dots = 0$ . If  $s \leq 2$ , then  $\tilde{H} = T^{-1}H(Tx)$  has the second form in the proof of theorem 3.4, so assume that  $s \geq 3$ . We distinguish three cases.

- $r \leq 1$ .  
Then  $\tilde{H}_2$  and  $\tilde{H}_3$  are independent linear forms. Hence we can take  $S \in \text{GL}_n(K)$  such that the first two rows of  $S^{-1}$  are  $\mathcal{J}\tilde{H}_2$  and  $\mathcal{J}\tilde{H}_3$ . By the chain rule,  $\mathcal{J}(\tilde{H}_2(Sx)) = e_1^t$  and  $\mathcal{J}(\tilde{H}_3(Sx)) = e_2^t$ , so  $\tilde{H}_2(Sx) = x_1$  and  $\tilde{H}_3(Sx) = x_2$ . Hence  $\tilde{H}(Sx) \in K[x_1, x_2]^n$  and  $S^{-1}\tilde{H}(Sx) = (TS)^{-1} \cdot H((TS)x)$  satisfies the first form in the proof of theorem 3.4.
- $r \geq 3$ .  
Since  $\text{rk } \mathcal{J}\tilde{H} = 2$ , the rows of  $\mathcal{J}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3)$  are dependent over  $K(x)$  and hence also over  $K[x]$ . By looking at leading homogeneous parts, we see that  $\text{rk } \mathcal{J}(\bar{H}_1, \bar{H}_2, \bar{H}_3) \leq 2$ , where  $\bar{H}_i$  is the leading homogeneous

part  $\tilde{H}_i$  for each  $i \leq 3$ . By [dB1, Th. 4.3.1], there exists linear forms  $p, q$  such that  $\tilde{H}_1, \tilde{H}_2, \tilde{H}_3$  are linearly dependent over  $K$  of  $p^2, pq$  and  $q^2$ . Furthermore,  $p$  and  $q$  are independent, and  $p^2, pq$  and  $q^2$  are in turn linearly dependent over  $K$  of  $\tilde{H}_1, \tilde{H}_2, \tilde{H}_3$ . Thus there exists an  $L \in \text{GL}_3(K)$  such that  $L(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3) = (p^2, pq, q^2)$ .

Take  $S \in \text{GL}_n(K)$  such that the first two rows of  $S^{-1}$  are  $\mathcal{J}p$  and  $\mathcal{J}q$ , in that order. Then  $L(\tilde{H}_1(Sx), \tilde{H}_2(Sx), \tilde{H}_3(Sx)) = (x_1^2, x_1x_2, x_2^2)$ . The minor determinants of size 2 of  $\mathcal{J}_{x_1, x_2}(x_1^2, x_1x_2, x_2^2)$  are  $2x_2^2, 4x_1x_2$  and  $2x_1^2$ , which are also linearly independent over  $K$ . It follows that

$$\det \mathcal{J}_{x_1, x_2, x_i}(L(\tilde{H}_1(Sx), \tilde{H}_2(Sx), \tilde{H}_3(Sx))) \neq 0$$

holds if  $i \geq 3$  and the last column of the Jacobian matrix on the left hand side is nonzero. Hence  $L(\tilde{H}_1(Sx), \tilde{H}_2(Sx), \tilde{H}_3(Sx)) \in K[x_1, x_2]^3$ . Since the first rows of its Jacobian are independent and  $L$  is invertible, we have  $\tilde{H}(Sx) \in K[x_1, x_2]^n$  as well. So  $S^{-1}\tilde{H}(Sx) = (TS)^{-1}H((TS)x)$  satisfies the first form in the proof of theorem 3.4.

- $r = 2$ .

If  $s \geq 4$ , then we can proceed as in the case  $r \leq 1$ , but with  $\tilde{H}_3$  and  $\tilde{H}_4$  instead of  $\tilde{H}_2$  and  $\tilde{H}_3$ . So assume that  $s = 3$ .

Since multiplication of the third row of  $\mathcal{J}\tilde{H}$  by  $2\tilde{H}_3$  does not change the rank of  $\mathcal{J}\tilde{H}$ , we have  $\text{rk } \mathcal{J}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3^2) \leq 2$ . Let  $\bar{H}_i$  be the leading homogeneous part of  $\tilde{H}_i$  for each  $i \leq 3$ . If  $\bar{H}_3^2$  is linearly independent over  $K$  of  $\bar{H}_1$  and  $\bar{H}_2$ , then we can proceed as in the case  $r \geq 3$  to obtain that  $\tilde{H}_i(Sx) \in K[x_1, x_2]$  for each  $i \neq 3$  and  $\tilde{H}_3(Sx)^2 \in K[x_1, x_2]$  for some  $S \in \text{GL}_n(K)$ . So  $S^{-1}\tilde{H}(Sx) = (TS)^{-1}H((TS)x)$  satisfies the first form in the proof of theorem 3.4 in that case.

So assume that  $\bar{H}_3^2$  is linearly dependent over  $K$  of  $\bar{H}_1$  and  $\bar{H}_2$ . Then we can choose  $T$  such that  $\tilde{H}_2 = \tilde{H}_3^2$ . If the linear part of  $\tilde{H}_2$  is dependent of  $\tilde{H}_3$ , then we can choose  $T$  such that even  $\tilde{H}_2 = \tilde{H}_3^2$ . Since  $s = 3$ , we see that  $\tilde{H} = T^{-1}H(Tx)$  has the third form in the proof of theorem 3.4 in that case.

So assume that the linear part of  $\tilde{H}_2$  is independent of  $\tilde{H}_3$ . Then  $\tilde{H}_2 - \tilde{H}_3^2$  and  $\tilde{H}_3$  are independent linear forms. Since  $\mathcal{J}(\tilde{H}_2 - \tilde{H}_3^2) = \mathcal{J}\tilde{H}_2 - 2\tilde{H}_3\mathcal{J}\tilde{H}_3$ , we can replace  $\tilde{H}_2$  by  $\tilde{H}_2 - \tilde{H}_3^2$  with affecting the Jacobian rank of  $\tilde{H}$ , and proceed as in the case  $r \leq 1$  to obtain that  $e_1^t$  and  $e_2^t$  are in the row space of  $\mathcal{J}\tilde{H}(Sx)$  for some  $S \in \text{GL}_n(K)$ . Hence  $\tilde{H}(Sx) \in K[x_1, x_2]^n$  and  $S^{-1}\tilde{H}(Sx) = (TS)^{-1}H((TS)x)$  satisfies the first form in the proof of theorem 3.4.  $\square$

## 4 Lemmas

The lemmas in this section are required for the proofs that the counterexamples in the next section are indeed counterexamples.

**Lemma 4.1.** *Let  $d \geq 1$  and  $a_1, a_2, \dots, a_{2d+2} \in K^n$  be pairwise independent. Suppose that for all  $j \geq \min\{3, d^2\}$  and all  $k$  with  $3 \leq k \leq d+2$ , the set  $\{a_j, a_k, a_{k+d}\}$  consist of two or three independent vectors (depending on whether  $j \in \{k, k+d\}$ ).*

*If*

$$\sum_{i=1}^{2d+2} \lambda_i (a_i^t x)^d = 0 \quad (5)$$

*for some  $\lambda_i \in K$ , not all zero, then  $\lambda_1 \lambda_2 \neq 0$ .*

*Proof.* Assume that (5) holds. If  $d = 1$ , then  $\lambda_1 \lambda_2 = 0$  implies that either  $a_1$  or  $a_2$  is dependent of  $a_3$  and  $a_4$ , which is a contradiction. Hence the following cases remain.

- $d = 2$ .

Since  $a_3, a_4$  and  $a_6$  are independent, we may assume without loss of generality that  $a_1, a_2, a_6$  are independent vectors. Consequently, there exists a  $b_1 \in K^n$  such that  $b_1^t a_1 = b_1^t a_6 = 0 \neq b_1^t a_2$ . Applying  $b_1^t \partial$  on (5) gives

$$\sum_{i=2}^5 \mu_i (a_i^t x)^1 = 0$$

where  $\mu_i = 2\lambda_i b_1^t a_i$  for all  $i$ . Since  $a_3, a_4, a_5$  are independent, we have  $\mu_2 \neq 0$ . Hence  $\lambda_2 \neq 0$ . In a similar manner,  $\lambda_1 \neq 0$  follows.

- $d > 2$ .

Since  $a_1$  and  $a_2$  are independent, we may assume without loss of generality that  $\lambda_3 \neq 0$ . Since  $a_3, a_{d+2}$  and  $a_{2d+2}$  are independent, there exists a  $b_2 \in K^n$  such that  $b_2^t a_{d+2} = b_2^t a_{2d+2} = 0 \neq b_2^t a_3$ . Applying  $b_2^t \partial$  on (5) gives

$$\sum_{i=1}^{d+1} \mu_i (a_i^t x)^{d-1} + \sum_{i=d+3}^{2d+1} \mu_i (a_i^t x)^{d-1} = 0$$

where  $\mu_i = d\lambda_i b_2^t a_i$  for all  $i$ . Since  $\mu_3 \neq 0$ , it follows by induction on  $d$  that  $\mu_1 \mu_2 \neq 0$ . Hence  $\lambda_1 \lambda_2 \neq 0$ .  $\square$

**Lemma 4.2.**

$$\sum_{i=0}^d (-1)^i \binom{d}{i} (x_1 + ix_3)^d = \sum_{i=0}^d (-1)^i \binom{d}{i} (x_2 + ix_3)^d \quad (6)$$

and

$$\sum_{i=0}^{d-1} \zeta_d^i (\zeta_d^i x_1 + x_2 + x_3)^d + \sum_{i=0}^{d-1} \zeta_d^i (\zeta_d^i x_1 + x_2 - x_3)^d = 2d^2 x_1^{d-1} x_2 \quad (7)$$

*Proof.* We first prove (6). Assume that (6) holds when we replace  $d$  by  $d-1$ . By substituting  $x_2 = x_1 + x_3$  on both side, we obtain

$$\begin{aligned} \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i} (x_1 + ix_3)^{d-1} &= \sum_{i=0}^{d-1} (-1)^i \binom{d-1}{i} (x_1 + (i+1)x_3)^{d-1} \\ &= - \sum_{i=1}^d (-1)^i \binom{d-1}{i-1} (x_1 + ix_3)^{d-1} \end{aligned}$$

By  $\binom{d}{i} = \binom{d-1}{i-1} + \binom{d-1}{i}$ , both sides combine to

$$0 = \sum_{i=0}^d (-1)^i \binom{d}{i} (x_1 + ix_3)^{d-1} = \frac{1}{d} \partial_1 \sum_{i=0}^d (-1)^i \binom{d}{i} (x_1 + ix_3)^d$$

Hence the left hand side of (6) is contained in  $K[x_3]$ . By a symmetry argument, (6) follows by induction on  $d$ .

By substituting  $x_2 = x_2 \pm x_3$  in

$$\sum_{i=0}^{d-1} \zeta_d^i (\zeta_d^i x_1 + x_2)^d = d^2 x_1^{d-1} x_2 \quad (8)$$

we get (7). So in order to prove (7), it suffices to show (8). This can be done as follows.

$$\begin{aligned} \sum_{i=0}^{d-1} \zeta_d^i (\zeta_d^i x_1 + x_2)^d &= \sum_{i=0}^{d-1} \zeta_d^i \sum_{j=0}^d \binom{d}{j} (\zeta_d^i x_1)^j x_2^{d-j} \\ &= \sum_{j=0}^d \binom{d}{j} x_1^j x_2^{d-j} \sum_{i=0}^{d-1} \zeta_d^{i(j+1)} \\ &= \binom{d}{d-1} x_1^{d-1} x_2 \sum_{i=0}^{d-1} 1 \\ &= d^2 x_1^{d-1} x_2 \quad \square \end{aligned}$$

**Lemma 4.3.** *To write  $x_1^{d-1} x_2$  as a linear combination of  $x_1^d$  and other  $d$ -th powers of linear forms, at least  $d$  such powers are necessary besides  $x_1^d$ .*

*Proof.* The case  $d = 1$  is easy, so let  $d \geq 2$  and suppose that  $x_1^{d-1} x_2$  can be written as a linear combination of  $x_1^d, (a_3^t x)^d, (a_4^t x)^d, \dots, (a_{d+1}^t x)^d$ . Assume without loss of generality that the vectors  $e_1, a_3, a_4, \dots, a_{d+1}$  are pairwise independent. By applying  $\partial_2$  on this linear combination, we obtain that

$$x_1^{d'} = \lambda_3 (a_3^t x)^{d'} + \lambda_4 (a_4^t x)^{d'} + \dots + \lambda_{d'+2} (a_{d'+2}^t x)^{d'}$$

where  $d' = d - 1$ . Take  $a_1 = e_1$  and take  $a_2$  independent of  $a_1, a_3, \dots, a_{d'+2}$ . Next, take  $a_i$  independent of  $a_1, a_2, \dots, a_{i-1}$  for all  $i$  with  $d' + 3 \leq i \leq 2d' + 2$ . Then lemma 4.1 with  $d$  replaced by  $d'$  gives a contradiction.  $\square$

## 5 Counterexamples

We start with giving counterexamples  $x + H$  to  $(\text{JC}^-) \Rightarrow (\text{JC})$  and  $(\text{JC}^+) \Rightarrow (*)$ , such that  $H$  is homogeneous of degree  $d \geq 3$  and  $d \geq 2$  respectively. With known techniques, these counterexamples can be improved to counterexamples of the form  $x + (Ax)^{*d}$ .

**Theorem 5.1.** *If  $n = 4$  and  $d \geq 3$ , then*

$$H = x_1^{d-3}(0, 0, x_2(x_1x_3 - x_2x_4), x_1(x_1x_3 - x_2x_4)) \quad (9)$$

*is a homogeneous counterexample of degree  $d$  to  $(\text{JC}^-) \Rightarrow (\text{JC})$ .*

*If  $n = 5$  and  $d \geq 2$ , then*

$$H = (0, 0, x_2^{d-1}x_4, x_1^{d-1}x_3 - x_2^{d-1}x_5, x_1^{d-1}x_4) \quad (10)$$

*is a homogeneous counterexample of degree  $d$  to  $(\text{JC}^+) \Rightarrow (*)$ .*

*Furthermore, there exist a power linear counterexample to  $(\text{JC}^-) \Rightarrow (\text{JC})$  for each  $d \geq 3$ , and a power linear counterexample to  $(\text{JC}^+) \Rightarrow (*)$  for each  $d \geq 2$ .*

*Proof.* Assume first that  $n = 4$  and  $H$  is as in (9). Since the components of  $H$  are composed of the invariants  $x_1, x_2, x_1x_3 - x_2x_4$  of  $x + H$ , we see that  $x + H$  is a quasi-translation, i.e.  $x - H$  is the inverse of  $x + H$ . One can compute that the trailing principal minor matrix of size 2 of  $(d-1)I_4 + (d-2)(\mathcal{J}H)|_{x=(1,0,0,0)} + (\mathcal{J}H)|_{x=(1,c,0,0)}$  equals

$$\begin{pmatrix} d-1+c & -c^2 \\ d-1 & d-1-c \end{pmatrix}$$

and that its determinant equals  $c^2(d-2) + (d-1)^2$ . So if we take  $c = \frac{d-1}{\sqrt{d-2}}i$ , then

$$\det \left( (d-1)I_4 + (d-2)(\mathcal{J}H)|_{x=(1,0,0,0)} + (\mathcal{J}H)|_{x=(1,c,0,0)} \right) = 0$$

which contradicts  $(\text{JC}^-) \Rightarrow (\text{JC})$ .

Assume next that  $n = 5$  and  $H$  is as in (10). Then one can compute that

$$\mathcal{J}H = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & b & 0 \\ * & * & a & 0 & -b \\ * & * & 0 & a & 0 \end{pmatrix}$$

The form on the right hand side does not change by substitution and adding copies of  $\mathcal{J}H$  with different substitutions, so  $\sum_{i=1}^n (\mathcal{J}H)|_{x=v_i}$  is nilpotent for all  $v_i \in K^n$ . This gives  $(\text{JC}^+)$ .

On the other hand,  $(\mathcal{J}H)|_{x_1=0} \cdot (\mathcal{J}H)|_{x_2=0}$  is a lower triangular matrix with diagonal  $(0, 0, x_1^{d-1}x_2^{d-1}, -x_1^{d-1}x_2^{d-1}, 0)$ , so  $(\mathcal{J}H)|_{x_1=0} \cdot (\mathcal{J}H)|_{x_2=0}$  is not

nilpotent. By [vdEH] and  $(\text{JC}^+)$ , we see that  $H$  is a counterexample to  $(\text{JC}^+) \Rightarrow (*)$ .

To obtain power linear counterexamples, we can use the concept of GZ-pairing in [GZ]. For that purpose, let  $H$  be any of the above two maps. By [GZ, Th. 1.3], there exists an  $N > n$  and an  $A \in \text{Mat}_N(K)$ , such that  $x + H$  and  $X + (AX)^{*d}$  are GZ-paired through matrices  $B \in \text{Mat}_{n,N}(K)$  and  $C \in \text{Mat}_{N,n}(K)$ , where  $X = (x_1, x_2, \dots, x_N)$ . Take  $M \in \text{Mat}_{N, N-n}(K)$  such that the columns of  $M$  form a basis of  $\ker B$  and define  $\bar{T} = (C \mid M)$ .

Then one can show that  $\bar{T}$  is as in the proof of [Che, Th. 2], with  $F = X + (AX)^{*d}$  and  $F_1 = (x + H, \dots)$ . Now one can use similar techniques as in the proof of theorem 3.1 to obtain that  $(AX)^{*d}$  is a counterexample as well as  $H$ , or use the following invariance results for GZ-pairing. The GZ-invariance of  $(\text{JC}^-)$  follows from [GZ, Th. 1.3 (9)] and that of  $(*)$  from [LDS, Th. 3 (2)]. The GZ-invariance of  $(\text{JC})$  and  $(\text{JC}^+)$  can be proved with techniques in the proof [GZ, Th. 2.4].  $\square$

*Example 5.2.* Let  $x = (x_1, x_2, \dots, x_5)$  and  $X = (x_1, x_2, \dots, x_{13})$ . Take  $H$  as in (10) and

$$G = (0, 0, (x_4 - x_1)^3, (x_4 + x_1)^3, x_4^3, (x_4 - x_2)^3, (x_4 + x_2)^3, \\ (x_3 - x_1)^3, (x_3 + x_1)^3, x_3^3, (x_5 - x_2)^3, (x_5 + x_2)^3, x_5^3)$$

Then  $\ker \mathcal{J}_x G$  is trivial and  $6H = BG$ , where

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & -1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

has full rank. Hence there exists a matrix  $C$  such that  $BC = I_5$ . Consequently,  $x + H$  and  $X + \frac{1}{6}G(BX)$  are GZ-paired through  $B$  and  $C$ .

In the next theorem, we give threedimensional counterexamples  $F = x + H$  to  $(*) \Rightarrow (**)$  and  $(**) \Rightarrow (***)$ , such that  $H$  is homogeneous of degree  $d \geq 3$ . The techniques in the proof of the previous theorem to get counterexamples of the form  $F = x + (Ax)^{*d}$  do not work, so we improve our counterexamples to that form by hand.

**Theorem 5.3.** Assume that  $d \geq 2$ , and either

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ \vdots \\ H_{d+2} \\ H_{d+3} \\ H_{d+4} \\ \vdots \\ H_{2d+2} \end{pmatrix} := \begin{pmatrix} 0 \\ \nu x_1^d \\ x_1^d - x_2^d \\ (x_1 + 2x_3)^d \\ \vdots \\ (x_1 + dx_3)^d \\ (x_2 + x_3)^d \\ (x_2 + 2x_3)^d \\ \vdots \\ (x_2 + dx_3)^d \end{pmatrix} \quad (11)$$

or

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \\ \vdots \\ H_{d+2} \\ H_{d+3} \\ H_{d+4} \\ \vdots \\ H_{2d+2} \end{pmatrix} := \begin{pmatrix} 0 \\ \nu x_1^d \\ x_1^{d-1}x_2 \\ (\zeta_d x_1 + x_2 + x_3)^d \\ \vdots \\ (\zeta_d^{d-1}x_1 + x_2 + x_3)^d \\ (x_1 + x_2 - x_3)^d \\ (\zeta_d x_1 + x_2 - x_3)^d \\ \vdots \\ (\zeta_d^{d-1}x_1 + x_2 - x_3)^d \end{pmatrix} \quad (12)$$

for some  $\nu \in K$ . Then  $2d + 2 \geq 6$ ,

$$d(x_1 + x_3)^d = H_3 + \sum_{i=2}^d (-1)^i \binom{d}{i} H_{i+2} - \sum_{i=1}^d (-1)^i \binom{d}{i} H_{i+d+2} \quad (13)$$

in case of (11) and

$$(x_1 + x_2 + x_3)^d = 2d^2 H_3 - \zeta_d H_4 - \zeta_d^2 H_5 - \cdots - \zeta_d^{2d-1} H_{2d+2} \quad (14)$$

in case of (12), and there exists a  $T \in \text{GL}_{2d+2}(K)$  such that  $T(H(T^{-1}x))$  is power linear in case  $n = 2d + 2$ .

If  $3 \leq n \leq 2d + 2$ , then  $H = (H_1, H_2, \dots, H_n)$  is of the form (\*) and we have the following.

- i) If  $H$  is of the form (\*\*), then  $c_1$  and  $c_2$  are independent linear combinations of  $e_1$  and  $e_2$ .
- ii)  $H$  is of the form (\*\*), if and only if either  $H$  is as in (11) or  $H$  is as in (12) with  $H_2 = 0 = d - 2$ .
- iii)  $H$  is of the form (\*\*\*), if and only if  $H$  is as in (11) with  $H_2 \neq 0$ .



*Proof.* Since  $\mathcal{J}H$  is lower triangular with zeroes on the diagonal, it follows from i) of theorem 2.3 that  $H$  is of the form (\*). By (6) and (7) in lemma 4.2, we get (13) and (14) respectively. So  $H$  is a linear triangularization of a power linear map in case  $n = 2d + 2$ .

In case of (11), set

$$a_{2+i}^t x := x_1 + ix_2 \quad a_{d+2+i}^t x := x_1 + ix_3$$

for  $i = 1, 2, \dots, d$ . In case of (12), set

$$a_{2+i}^t x := \zeta_d^{i-1} x_1 + x_2 + x_3 \quad a_{d+2+i}^t x := \zeta_d^{i-1} x_1 + x_2 - x_3$$

for  $i = 1, 2, \dots, d$ . Then  $H_i = (a_i^t x)^d$  for all  $i \geq 4$  and the left hand side of (13) or (14) respectively is a multiple of  $(a_3^t x)^d$ . Hence the linear span  $S$  of  $H_3, H_4, \dots, H_n$  is contained in that of  $(a_3^t x)^d, (a_4^t x)^d, \dots, (a_{2d+2}^t x)^d$ .

- i) *Claim.* If  $\mu^t H$  and  $\mu_2 H_2$  are both linearly dependent of the same power of a linear form in  $x_1$  and  $x_2$  for some  $\mu \in K^n$ , then  $\mu$  is a linear combination of  $e_1$  and  $e_2$ .

To prove the claim, assume that  $\mu^t H$  and  $\mu_2 H_2$  are as above. Then there exists a nontrivial linear combination  $a_2$  of  $e_1$  and  $e_2$  such that both  $\mu^t H$  and  $\mu_2 H_2$  are linearly dependent of  $(a_2^t x)^d$ . On account of  $H_1 = 0$ , we have

$$0x_4^d + (\mu_2 H_2 - \mu^t H) + \mu_3 H_3 + \mu_4 H_4 + \dots + \mu_n H_n = 0$$

Take  $a_1 = e_4$ . By (13) and (14) respectively, there exist a  $\lambda \in K^{2d+2}$  with  $\lambda_1 = 0$ , such that

$$\lambda_1 (a_1^t x)^d + \lambda_2 (a_2^t x)^d + \lambda_3 (a_3^t x)^d + \dots + \lambda_{2d+2} (a_{2d+2}^t x)^d = 0$$

But additionally, (13) and (14) respectively tell us that  $\lambda_i \neq 0$  for some  $i \geq 3$  in case  $\mu_i \neq 0$  for some  $i \geq 3$ . By lemma 4.1 and  $\lambda_1 = 0$ , we have  $\lambda = 0$ . So  $\mu$  is a linear combination of  $e_1$  and  $e_2$  and the claim has been proved.

Suppose that  $H$  is of the form (\*\*). Since  $c_j^t b_i = 0$  for all  $i \geq j \geq 1$ , we have

$$c_1^t H = c_1^t \sum_{i=1}^{n-1} (c_i^t x)^d b_i = \sum_{i=1}^{n-1} c_1^t b_i (c_i^t x)^d = 0 \quad (15)$$

thus  $c_1$  is a linear combination of  $e_1$  and  $e_2$  on account of the above claim. Using (15) again, we see that  $c_1$  is linearly dependent of  $e_1$  if  $H_2 \neq 0$ . Hence  $(c_1^t x)^d$  and  $H_2$  are linearly dependent of the same power of a linear form in  $x_1$  and  $x_2$ . By using  $c_j^t b_i = 0$  for all  $i \geq j \geq 1$  again, we obtain that

$$c_2^t H = c_2^t \sum_{i=1}^{n-1} (c_i^t x)^d b_i = \sum_{i=1}^{n-1} c_2^t b_i (c_i^t x)^d = c_2^t b_1 (c_1^t x)^d \quad (16)$$

It follows from the above claim that  $c_2$  is a linear combination of  $e_1$  and  $e_2$  as well.

Suppose that  $c_1$  and  $c_2$  are dependent. Then there exist a nontrivial linear combination  $a_2$  of  $e_1$  and  $e_2$  such that both  $c_1$  and  $c_2$  are dependent of  $a_2$ . Using the claim with  $\mu_1 = \mu_2 = 0$ , and  $\mu^t H = 0$  and  $\mu^t H = (a_2^t x)^d$  respectively, we obtain  $\dim S = n - 2$  and  $(a_2^t x)^d \notin S$ .

The space  $S^*$  generated by  $(c_1^t x)^d, (c_2^t x)^d, \dots, (c_{n-1}^t x)^d$ , which contains  $S$ , is generated by  $n - 2$  powers of linear forms, namely  $(a_2^t x)^d, (c_3^t x)^d, (c_4^t x)^d, \dots, (c_{n-1}^t x)^d$ . Hence  $S^* \supseteq S$  and  $\dim S^* \leq n - 2 = \dim S$ . It follows that  $S = S^*$ . Since  $(a_2^t x)^d \notin S = S^*$ , we have  $c_1 = c_2 = 0$  and  $\dim S^* < n - 2$ . This contradicts  $S = S^*$  and  $\dim S = n - 2$ , so  $c_1$  and  $c_2$  are independent.

ii) If  $H$  is as in (11), then we can take

$$\begin{aligned} c_1 &= e_1 & b_1 &= \nu e_2 + e_3 \\ c_2 &= e_2 & b_2 &= e_3 \\ c_i &= a_{i+1} & b_i &= e_{i+1} \end{aligned}$$

for all  $i > 2$ , which shows that  $H$  is of the form (\*\*). If  $H$  is as in (12) with  $H_2 = 0 = d - 2$ , then we can take

$$\begin{aligned} c_1 &= e_1 + e_2 & b_1 &= \frac{1}{4}e_3 \\ c_2 &= e_1 - e_2 & b_2 &= -\frac{1}{4}e_3 \\ c_i &= a_{i+1} & b_i &= e_{i+1} \end{aligned}$$

for all  $i > 2$ , which shows that again  $H$  is of the form (\*\*).

Conversely, suppose that  $H$  is as in (12) and of the form (\*\*). By lemma 4.3, at least  $d$  powers of linear forms are necessary to write  $H_2$  and  $H_3$  as linear combinations of them in case  $H_2 = 0$ , and at least  $d + 1$  such powers otherwise. Now assume that  $H_2 \neq 0$  or  $d \geq 3$ . Then there are at least 3 powers of linear forms necessary to write  $H_2$  and  $H_3$  as linear combinations of them. Hence there exist a linear combination  $h$  of  $H_2$  and  $H_3$  which is not a linear combination of  $(c_1^t x)^d$  and  $(c_2^t x)^d$ .

Since  $\dim S^* \leq n - 1 < n$ , there exists a nonzero  $\mu \in K^n$  such that

$$\mu_1(c_1^t x)^d + \mu_2(c_2^t x)^d + \mu_3 h + \mu_4 H_4 + \mu_5 H_5 + \dots + \mu_n H_n = 0$$

By applying  $\partial_3$  on both sides, we get

$$\begin{aligned} d\mu_4(a_4^t x)^{d'} + \dots + d\mu_{d+2}(a_{d+2}^t x)^{d'} - d\mu_{d+3}(a_{d+3}^t x)^{d'} \\ - d\mu_{d+4}(a_{d+4}^t x)^{d'} - \dots - d\mu_{2d+2}(a_{2d+2}^t x)^{d'} = 0 \end{aligned} \quad (17)$$

where  $d' = d - 1$  and  $\mu_{n+1} = \mu_{n+2} = \dots = \mu_{2d+2} = 0$ . Take  $a'_1 = e_4$  and  $a'_2 = a_{d+3}$ . Additionally set  $a'_{i+1} = a_{i+2}$  and  $a'_{i+d} = a_{i+d+2}$  for all  $i$  with  $2 \leq i \leq d$ . By lemma 4.1 with  $d$  and  $a$  replaced by  $d'$  and  $a'$  respectively, we get  $\mu_4 = \mu_5 = \dots = \mu_{2d+2} = 0$ . Hence  $h$  is a linear combination of  $(c_1^t x)^d$  and  $(c_2^t x)^d$ . Contradiction, so  $H$  is not of the form (\*\*).

- iii) Assume first that  $H_2 \neq 0$ . If  $H$  is as in (11), then we can take the  $c_j$ 's and the  $b_i$ 's as in ii), and we have (\*\*). If  $H$  is as in (12), then by ii),  $H$  is not of the form (\*\*) and hence neither of the form (\*\*).

Assume next that  $H_2 = 0$  and that  $H$  is of the form (\*\*). By  $H_1 = H_2 = 0$  and the fact that  $c_2$  is dependent of  $e_1$  and  $e_2$ , we have  $c_2^t H = 0$ . Consequently,  $c_2^t b_1 = 0$  on account of (16). By definition of (\*\*), we have  $c_1^t b_i = c_2^t b_i = 0$  for all  $i$ . Since  $c_1$  and  $c_2$  are independent, we have a contradiction with the independence of the  $b_i$ 's.  $\square$

We can make non-homogeneous variants of (9) and (10) as follows. In (9), we can replace  $x_2$  by 1, remove  $H_2$ , and replace  $x_{i+1}$  by  $x_i$  for all  $i \geq 1$ . In (10), we can replace  $x_2^{d-1}$  by  $x_1^{d-2}$ , remove  $H_2$ , and replace  $x_{i+1}$  by  $x_i$  for all  $i \geq 1$ . In this manner, we get rid of the second coordinate, such that both the dimension and the Jacobian rank respectively decrease by one, in return for abandoning homogeneity, just as with most of theorem 3.2 with respect to theorem 3.1.

The maps  $H = (0, x_1^d - x_1^{d-1})$  and  $H = (0, 0, x_1^d - x_1^{d-1})$  are additional non-homogeneous counterexamples to  $(*) \Rightarrow (**)$  and  $(**) \Rightarrow (***)$  respectively. By comparing the counterexamples with the positive results of theorems 3.1, 3.2 and 3.4, we get the following four questions.

The first two questions are whether (JC) implies (JC<sup>+</sup>) in general and whether (JC<sup>+</sup>) implies (\*) in dimension three. In case  $H$  is homogeneous, then the questions are whether (JC) implies (JC<sup>+</sup>) in general and whether (JC<sup>+</sup>) implies (\*) in dimension four, which are the last two questions. By theorems 3.1 and 3.2, the last and the second question respectively have an affirmative answer when the degree is at most three.

## References

- [dB1] Michiel de Bondt. *Homogeneous Keller maps*. PhD thesis, University of Nijmegen, Toernooiveld, 6525 ED Nijmegen, The Netherlands, July 7 2009. Directed by A.R.P. van den Essen.
- [dB2] Michiel de Bondt. The strong nilpotency index of a matrix. arXiv:1203.6615, 2012.
- [dBvdE] Michiel de Bondt and Arno van den Essen. The Jacobian conjecture: linear triangularization for homogeneous polynomial maps in dimension three. *J. Algebra*, 294(1):294–306, 2005.
- [Che] Charles Ching-An Cheng. Power linear Keller maps of rank two are linearly triangularizable. *J. Pure Appl. Algebra*, 195(2):127–130, 2005.
- [Dru] Ludwik M. Drużkowski. An effective approach to Keller's Jacobian conjecture. *Math. Ann.*, 264(3):303–313, 1983.
- [vdE] Arno van den Essen. *Polynomial automorphisms and the Jacobian conjecture*, volume 190 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2000.

- [vdEH] Arno van den Essen and Engelbert Hubbers. Polynomial maps with strongly nilpotent Jacobian matrix and the Jacobian conjecture. *Linear Algebra Appl.*, 247:121–132, 1996.
- [GZ] Gianluca Gorni and Gaetano Zampieri. On cubic-linear polynomial mappings. *Indag. Math. (N.S.)*, 8(4):471–492, 1997.
- [GdBDS] Hongbo Guo, Michiel de Bondt, Xiankun Du, and Xiaosong Sun. Polynomial maps with invertible sums of Jacobian matrices and directional derivatives. *Indag. Math. (N.S.)*, 23(3):256–268, 2012.
- [Kel] Ott-Heinrich Keller. Ganze Cremona-Transformationen. *Monatsh. Math. Phys.*, 47(1):299–306, 1939.
- [LDS] Dayan Liu, Xiankun Du, and Xiaosong Sun. Quadratic linear Keller maps of nilpotency index three. *Linear Algebra Appl.*, 429(1):12–17, 2008.
- [MO] Gary H. Meisters and Czesław Olech. Strong nilpotence holds in dimensions up to five only. *Linear and Multilinear Algebra*, 30(4):231–255, 1991.
- [TdB] He Tong and Michiel de Bondt. Power linear Keller maps with ditto triangularizations. *J. Algebra*, 312(2):930–945, 2007.